ГЕОДЕЗИЯ

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MATRIX GENERALIZATION OF THE PROBLEM OF 3D COORDINATE TRANSFORMATION IN SATELLITE CONSTRUCTIONS

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The article considers the problem of generalizing the 3D Helmert transformation by 7 parameters and differential equations of the 1st and 2nd types of spheroidal geodesy. A formula is obtained in matrix form, which includes, as special cases, almost all types of coordinate transformations used in satellite geodesy, as well as several new types.

Keywords: *spheroidal geodesy, seven parameter Helmert transformation, differential equations of the 1st and 2nd kind.*

Introduction. When processing satellite measurements to fix a point in space, quite a lot of different coordinate systems (CS) are currently used. Obviously, the task of recalculating the coordinates of points from the current system to the system required by the customer is quite relevant. Recalculation of coordinates in satellite networks can be divided into Cartesian (Helmert-type models), geodetic (Molodensky-type models) and combination [1].

Recalculation of coordinates is divided into transformation - coordinate operations using exact formulas without changing the datum, and transformation - coordinate operations in which the elements of recalculation are obtained from previous calculations and imply (not necessarily) a change in the datum.

Main part. The basis for the spatial transformation of Cartesian coordinates is the well-known linear movements in space, which in the most general form are a composition of rotations around three coordinate axes, shifts along three coordinate axes and scaling along three coordinate axes. In geodesy it is used in the form of Helmert's conformal transformation (similarity transformation) in which the shape and angles between lines are preserved. In this case, the scales along the axes are the same. Then, during transformation, the Cartesian rectangular coordinates of point $A(x, y, z)$ in CS A, goes to point $B(x', y', z')$ in CS B, by rotating around 3 axes in space, shifting along 3 axes and the same scaling factor.

Rotations around the X axis by an angle α, around the Y axis by an angle β, around the Z axis by an angle γ are determined, respectively, by rotation matrices in three-dimensional space $R_X(\alpha)$, $R_Y(\beta)$, $R_Z(\gamma)$. If we choose the Cartesian geocentric system as the spatial coordinate system, then two types of rotation are distinguished:

– *clockwise* rotations, when looking at the center of the coordinate system from the positive direction of the corresponding axis, are represented in matrix form as

$$R_X(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix},$$

around the X axis at an angle α,

$$R_Y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ \sin(\beta) & 0 & \cos(\beta) \end{bmatrix} = \begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix}, \qquad (1)$$

around the Y axis at an angle β,

$$R_Z(\gamma) = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

around the Z axis at an angle γ.

This representation of rotation is called coordinate frame (CF) - rotation of a point and is taken by us as the main one.

– *counterclockwise* rotations, when looking at the center of the coordinate system from the positive direction of the corresponding axis, are represented in matrix form as

$$R_X(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c} & -\mathrm{s} \\ 0 & \mathrm{s} & \mathrm{c} \end{bmatrix},$$

around the *X* axis at an angle *α*,

$$R_Y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} = \begin{bmatrix} \mathrm{c} & 0 & \mathrm{s} \\ 0 & 1 & 0 \\ -\mathrm{s} & 0 & \mathrm{c} \end{bmatrix}, \quad (2)$$

around the *Y* axis at an angle *β*,

$$R_Z(\gamma) = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c} & -\mathrm{s} & 0 \\ \mathrm{s} & \mathrm{c} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

around the *Z* axis at an angle *γ*.

This representation of rotation is called *vector position (VP) – vector rotation* [2].

Then the total rotation matrix *R* is usually assembled in the following two forms: Helmert rotation model of the 1st kind

$$R = R_{XYZ}(\alpha\beta\gamma) = R_Z(\gamma) \cdot R_Y(\beta) \cdot R_X(\alpha)\,; \quad (3)$$

$$R = R_{ZYX}(\gamma\beta\alpha) = R_X \cdot (\alpha) \cdot R_Y(\beta) \cdot R_Z(\gamma)\,.$$

Helmert rotation model of the 2nd kind.

Scaling along the *X* axis is by the value *mx*, along the *Y* axis is by the value *my*, along the *Z* axis by the value *mz*. In matrix form it can be represented as

$$M = \begin{bmatrix} m_X & 0 & 0 \\ 0 & m_Y & 0 \\ 0 & 0 & m_Z \end{bmatrix}. \quad (4)$$

Under the conformal Helmert transformation, *mx* = *my* = *mz* = *m*.

Translation along the *X* axis is by the value t_x along the *Y* axis is by the value t_y, along the *Z* axis by the value t_z. In matrix form it can be represented as a vector

$$t = \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix}. \quad (5)$$

In its full form, the transformation of a point from system *A* to system *B* in matrix representation is as follows

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_B = M \cdot R \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_A + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}. \quad (6)$$

Transformations of this type are called complete spatial linear transformation, or 9-parameter transformation. Then the exact conformal Helmert transformation is as follows

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_B = m \cdot R \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_A + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}. \quad (7)$$

If we multiply three partial rotations according to the first Helmert model in the form of a rotation of a point (the matrices rotate the point clockwise), and look from the positive end of the coordinate axis to the center of the CS, we obtain an expanded representation of the total rotation matrix R in the form

$$R = R_Z(\gamma)\cdot R_Y(\beta)\cdot R_X(\alpha) = \begin{bmatrix} \cos(\beta)\cdot\cos(\gamma) & \cos(\beta)\cdot\sin(\gamma) & -\sin(\beta) \\ \sin(\alpha)\cdot\sin(\beta)\cdot\cos(\gamma)-\cos(\alpha)\cdot\sin(\gamma) & \sin(\alpha)\cdot\sin(\beta)\cdot\sin(\gamma)+\cos(\alpha)\cdot\cos(\gamma) & \sin(\alpha)\cdot\cos(\beta) \\ \cos(\alpha)\cdot\sin(\beta)\cdot\cos(\gamma)+\sin(\alpha)\cdot\sin(\gamma) & \cos(\alpha)\cdot\sin(\beta)\cdot\sin(\gamma)-\sin(\alpha)\cdot\cos(\gamma) & \cos(\alpha)\cdot\cos(\beta) \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}. \quad (8)$$

In geodetic practice, when transforming coordinate systems on the earth's surface, the angles of rotation are very small (up to 1–2 arc seconds), the scale m along the axes differs very little from one (of the order of several ppm). Then, considering that

– the sines of small angles are equal to the angle itself in radians;
– the cosines of small angles tend to one;
– the product of the sines of small angles – a very small angle, which is taken equal to zero;
– the product of cosines of small angles is taken equal to one;
– the second order of smallness when multiplied by small and identical scales is negligible;
– the scale factor m differs very slightly from one and can be represented as $m = (1 + s)$.

The transformation formula will take the form

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_B = (1+s)\cdot\begin{bmatrix} 1 & w_Z & -w_Y \\ -w_Z & 1 & w_X \\ w_Y & -w_X & 1 \end{bmatrix}\cdot\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_A + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix} = m\cdot R_1\cdot\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_A + t \to K_B = m\cdot R_1\cdot K_A + t \quad (9)$$

or without taking into account the second order of smallness in the non-diagonal terms of the matrix (not always acceptable in terms of accuracy)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_B = \begin{bmatrix} (1+s) & w_Z & -w_Y \\ -w_Z & (1+s) & w_X \\ w_Y & -w_X & (1+s) \end{bmatrix}\cdot\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_A + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix} = R_2\cdot\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_A + t = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} s & w_Z & -w_Y \\ -w_Z & s & w_X \\ w_Y & -w_X & s \end{bmatrix}\right)\cdot\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_A + \begin{bmatrix} t_X \\ t_Y \\ t_Z \end{bmatrix} \to K_B = \left(E + R_3\right)\cdot K_A + t, \quad (10)$$

which is called *the Bursch-Wolff-2 transformation model* [2]. Obviously, the values of angles w_i are expressed in radians. In the classical Bursch-Wolff-1 model, the signs in the rotation angles are reversed.

Such transformation often referred to the 7-parameter *Helmert* transformation is used most often in higher geodesy and satellite positioning. The reverse transformation of coordinate system B into coordinate system A is as follows

$$K_A = \left(\frac{1}{m}\right)\cdot R_1^{-1}\cdot\left(K_B - t\right) = \left(1-s\right)\cdot R_1^T\cdot K_B - t. \quad (11)$$

The second expression in (11) does not take into account the second order of smallness when producing small numbers and can sometimes give unsatisfactory calculation accuracy.

Quite often in geodetic practice it is necessary to determine the coordinate increment from transformation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_B - \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_A = K_B - K_A = \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}_{AB} = \Delta K_{AB}. \quad (12)$$

Then from formulas (9) and (12) we have

$$\begin{aligned} K_B - K_A &= \Delta K_{AB} = \\ &= m \cdot R_1 \cdot K_A + t - K_A = , \\ &= \left(m \cdot R_1 - E\right) \cdot K_A + t \end{aligned} \tag{13}$$

where E is the identity matrix of size 3×3.

When transforming geodetic coordinates, the coordinates of point A, $(B, L, H)_A$ on ellipsoid 1, go to point A', $(B', L', H')_{A'}$ on ellipsoid 2 based on the increment of geodetic coordinates, as [1]

$$\begin{aligned} B' &= B + \Delta B; \\ L' &= L + \Delta L; \\ H' &= H + \Delta H. \end{aligned} \tag{14}$$

In the most general case, a problem of this kind is formulated as follows. On ellipsoid 1 there is a point with geodetic coordinates *(B, L, H)* in the coordinate system *A*. The value *H*, equal to the deviation of the point from the model ellipsoid along the normal, may be absent in the case when the point belongs to the ellipsoid, $H = 0$. Point 2 is obtained in relation to the first point by small rotations *(wx, wy, wz)*, shift by vector $(\Delta x, \Delta y, \Delta z)$, scaling *m* axes of the CS *A*, as well as changing the parameters of the ellipsoid when the point moves from ellipsoid 1 to ellipsoid 2 It is required to find the change in geodetic coordinates and the geodetic coordinates of point 2 *(B, L, H)*. To solve the problem, we find the complete differentials of the equations of connection between geodetic coordinates *(B, L, H)* and geocentric Cartesian coordinates *(x, y, z)* on an ellipsoid with characteristics a, e^2

$$\begin{cases} x = (N + H) \cdot \cos(B) \cdot \cos(L); \\ y = (N + H) \cdot \cos(B) \cdot \sin(L); \\ z = (N \cdot (1 - e^2) + H) \cdot \sin(B), \end{cases}$$

in all respects: B, L, H, a, e^2

$$\begin{aligned} dx &= \frac{\partial X}{\partial B} \cdot dB + \frac{\partial X}{\partial L} \cdot dL + \frac{\partial X}{\partial H} \cdot dH + \frac{\partial X}{\partial a} \cdot da + \frac{\partial X}{\partial e^2} \cdot de^2; \\ dy &= \frac{\partial Y}{\partial B} \cdot dB + \frac{\partial Y}{\partial L} \cdot dL + \frac{\partial Y}{\partial H} \cdot dH + \frac{\partial Y}{\partial a} \cdot da + \frac{\partial Y}{\partial e^2} \cdot de^2; \\ dz &= \frac{\partial Z}{\partial B} \cdot dB + \frac{\partial Z}{\partial L} \cdot dL + \frac{\partial Z}{\partial H} \cdot dH + \frac{\partial Z}{\partial a} \cdot da + \frac{\partial Z}{\partial e^2} \cdot de^2, \end{aligned} \tag{15}$$

which we will place in the table 1.

Table 1. – The complete differentials of the equations of connection between geodetic coordinates and geocentric Cartesian coordinates

derivatives by elements / coordinates	B	L	H	a	e^2
$x = (N+H)\cdot\cos(B)\cdot\cos(L)$	$-(M+H)\cdot\sin(B)\cdot\cos(L)$	$-(N+H)\cdot\cos(B)\cdot\sin(L)$	$\cos(B)\cdot\cos(L)$	$\frac{N}{a}\cdot\cos(B)\cdot\cos(L)$	$k\cdot\cos(B)\cdot\cos(L)$
$y = (N+H)\cdot\cos(B)\cdot\sin(L)$	$-(M+H)\cdot\sin(B)\cdot\sin(L)$	$(N+H)\cdot\cos(B)\cdot\cos(L)$	$\cos(B)\cdot\sin(L)$	$\frac{N}{a}\cdot\cos(B)\cdot\sin(L)$	$k\cdot\cos(B)\cdot\sin(L)$
$z = (N\cdot(1-e^2)+H)\cdot\sin(B)$	$(M+H)\cdot\cos(B)$	0	$\sin(B)$	$\frac{N}{a}\cdot(1-e^2)\cdot\sin(B)$	$\left(k\cdot(1-e^2)-N\right)\cdot\sin(B)$

Matrix A (columns B, L, H) Matrix B (columns a, e^2)

$$k = \left(\frac{N^2}{a^2} - 1\right) \cdot \frac{N}{2 \cdot e^2}$$

Considering the smallness of the measurements, we replace the differentials d with finite increments Δ. Then the general form of the change in Cartesian coordinates ΔD in matrix form will be

$$\Delta D = A \cdot \Delta G + B \cdot \Delta E , \tag{16}$$

where $\Delta D = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$ – small changes in Cartesian rectangular coordinates;

$\Delta G = \begin{bmatrix} \Delta B \\ \Delta L \\ \Delta H \end{bmatrix}$ – small changes in geodetic coordinates;

$\Delta E = \begin{bmatrix} \Delta a \\ \Delta e^2 \end{bmatrix}$ – small changes in the semi-major axis a of the ellipsoid and the second eccentricity e^2 when moving from ellipsoid 1 to ellipsoid 2;

$\Delta a = a_2 - a_1;$
$\Delta e^2 = e_2^2 - e_1^2.$

Matrices A and B are from the table of derivatives of communication equations.
It is sometimes convenient to represent matrix A in one of the following forms

$$A = \begin{bmatrix} -\sin(B)\cdot\cos(L) & -\sin(L) & \cos(B)\cdot\cos(L) \\ -\sin(B)\cdot\sin(L) & \cos(L) & \cos(B)\cdot\sin(L) \\ \cos(B) & 0 & \sin(B) \end{bmatrix} \cdot \begin{bmatrix} M+H & 0 & 0 \\ 0 & (N+H)\cdot\cos(B) & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$
$$= R3z(\pi - L)\cdot R2y(\pi/2 - B)\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot T = A_0 \cdot T.$$

Now, let's equate the increments of coordinates, or their change ΔD obtained on the basis of formulas (13) and (16). As a result, we have

$$\Delta D = A \cdot \Delta G + B \cdot \Delta E = (m \cdot R_1 - E)\cdot K_A + t, \tag{17}$$

a generalized system of equations, which includes all known transformations as special cases, and a number of new cases. To use it, you simply need to calculate the values of matrices A and B from (15) and table 1, and the value of matrix R_1 from formula (9).

Let us consider the most used special cases of formula (17).

1. The classical Molodensky transformation (14) from CS A on ellipsoid 1 to CS B on ellipsoid 2 in geodetic coordinates can be represented as

$$\Delta G = A^{-1} \cdot \left(\Delta D - B \cdot \Delta E\right), \tag{18}$$

if the change in the ellipsoid ΔE and the change in Cartesian coordinates are known ΔD;

$$\Delta G = A^{-1} \cdot \left(\left(m \cdot R_1 - E\right)\cdot K_A + t - B \cdot \Delta E\right), \tag{19}$$

if the change in the ellipsoid ΔE and 7 Helmert parameters are known;

$$\Delta G = A^{-1} \cdot \Delta D, \tag{20}$$

if there is no transition to another ellipsoid and the change in Cartesian coordinates is known ΔD;

$$\Delta G = A^{-1} \cdot \left(\left(m \cdot R_1 - E \right) \cdot K_A + t \right), \tag{21}$$

if there is no transition to another ellipsoid and 7 Helmert parameters are known;

$$\Delta G = -A^{-1} \cdot B \cdot \Delta E, \tag{22}$$

unless there is a transition to another ellipsoid.

2. The change in Cartesian rectangular coordinates with known changes in geodetic coordinates on one ellipsoid will have the form

$$\Delta D = A \cdot \Delta G, \tag{23}$$

and when moving to another ellipsoid

$$\Delta D = B \cdot \Delta E. \tag{24}$$

It should be considered that changes in Cartesian coordinates must be on the order of ten meters, and in angular coordinates - on the order of 1–2 seconds, so that the formulas produce a fairly accurate result, since in essence they are differential. In satellite positioning practice, these conditions are generally met. If the changes are large enough, then you need to use a sequence of actions based on coordinate transformation to solve the problem.

It is not difficult to notice that the coefficient matrices in (17) are Jacobian matrices, which makes it possible to use the resulting formulas when adjusting geodetic constructions in different coordinate systems and when assessing the accuracy of transformation.

Conclusion. General formulas are obtained that makes it possible to obtain small changes in the parameters of one system when moving to another coordinate system in matrix form. Formulas can be used to calculate small shifts in a coordinate system, compile correction equations for adjustment, and evaluate accuracy when moving from one coordinate system to another.

REFERENCE

1. Morozov, V.P. (1979). *Kurs sferoidicheskoi geodezii (Izd. 2, pererab. i dop.)*. Moscow: Nedra. (In Russ.).
2. Soler, T. (1976). On differential transformations between Cartesian and curvilinear (geodetic) coordinates. *Report number: Department of Geodetic Science*, (236).

ЛИТЕРАТУРА

1. Морозов В.П. Курс сфероидической геодезии. – Изд. 2, перераб. и доп. – М.: Недра, 1979. – 296 с.
2. Soler T. On differential transformations between Cartesian and curvilinear (geodetic) coordinates // Report number: Department of Geodetic Science. – January 1976. – Rep. № 236.

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МАТРИЧНОЕ ОБОБЩЕНИЕ ЗАДАЧИ 3D-ТРАНСФОРМИРОВАНИЯ КООРДИНАТ В СПУТНИКОВЫХ ПОСТРОЕНИЯХ

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В статье рассмотрена задача обобщения 3D-трансформирования Гельмерта по 7 параметрам и дифференциальных уравнений 1 и 2 рода сфераидической геодезии. Получена формула в матричном виде, включающая в себя как частные случаи практически все виды трансформирования координат, используемые в спутниковой геодезии, а также несколько новых видов.

Ключевые слова: *сфероидическая геодезия,7 параметрическое преобразование Гельмерта, дифференциальные уравнения 1 и 2 рода.*