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# SOLUTION OF A MULTIDIMENSIONAL INTEGRAL EQUATION OF THE FIRST KIND WITH THE BESSEL – KLIFFORD FUNCTION IN THE KERNEL OVER A PYRAMIDAL DOMAIN

***O. SKOROMNIK, T. ALEKSANDROVICH***
***(Polotsk State University)***

*The multidimensional integral equation of the first kind with the Bessel – Klifford function in the kernel over the special bounded pyramidal domain in Euclidean space is considered. The interest in such equations is caused by their applications to the problems on the reflection of waves on a rectilinear boundary and on a supersonic flow around spatial corners. Ya. Tamarkin obtained a well-known classical result on the solvability of the Abel integral equation in the space $L_1(a,b)$ of integrable functions on a finite interval $[a,b]$ of the real line. By Tamarkin's method the solution of the investigating equation in the closed form is established, and necessary and sufficient conditions for its solvability in the space of summable functions are given. The results generalize the well know findings for the multi-dimensional Abel type integral equation and the corresponding one-dimensional hypergeometric equations.*

**1. Introduction.** One – dimensional integral equations of the first kind, which generalize the classical Abel integral equation and contain the Gauss hypergeometric function, the Legendre function, the Kummer hypergeometric function and other special functions in their kernels, have been studied by many authors (see a survey of results and a bibliography in [1, Section 39]. Such equations arise in studying boundary value problems for equations of the hyperbolic and mixed type with boundary conditions containing generalized fractional integrals and derivatives [2]. In most papers, methods for studying Abel-type equations with hypergeometric functions, the Legendre function in kernels were based on representing the integral operators of these equations as compositions of fractional integration operators with power weights and using well-known properties of fractional integrals. In this way, sufficient solvability conditions for the integral equations under consideration in certain classes of functions and the solution of such equations in quadratures were obtained [1, sections 35.1, 35.2, 37.1].

The investigation of necessary and sufficient solvability conditions for equations mentioned above is a more difficult task. Ya. Tamarkin obtained a well-known classical result on the solvability of the Abel integral equation in the space $L_1(a,b)$ of integrable functions on a finite interval $[a,b]$ of the real line [1, Theorem 2.1]. In [3], a similar result was obtained for the multidimensional Abel-type integral equations over a special bounded pyramidal domain in Euclidean space. The interest in such equations is caused by their applications to the problems on the reflection of waves on a rectilinear boundary [4, p. 48; 5] and on a supersonic flow around spatial corners [6] (see also [1, Sections 25.1 and 28.4]).

Tamarkin's method [7; 8] was applied to obtain necessary and sufficient solvability conditions in the space of summable functions for a one-dimensional and multidimensional Abel-type integral equations with the Gauss hypergeometric function over a pyramidal domain.

In [9] the closed-form solutions of more general integral equations over pyramidal domains were obtained and necessary and sufficient solvability conditions in the space of summable functions were established. The analogical results were also received for the multidimensional integral equations of the first kind with the Legendre and Kummer functions in the kernels over pyramidal domains in [10; 11].

The target of this paper is to continue the aforementioned study. We give a closed – form solution to an integral equation with the Bessel-Klifford function over a pyramidal domain and investigate its solvability in the space of integrable functions.

**2. Preliminary data.** We use the following notations (see [1, Section 28.4]). By $\mathrm{N}=\{1,2,...\}$ we denote the set of positive integers, $\mathrm{N}_0=\mathrm{N}\bigcup\{0\}$, and $\mathrm{R}^n$ – Euclidean $n$-space. For vectors $\mathrm{x}=(x_1,x_2,...,x_n)\in\mathrm{R}^n$ and $\mathrm{t}=(t_1,t_2,...,t_n)\in\mathrm{R}^n$, $\mathrm{x}\cdot\mathrm{t}=\sum_{k=1}^{n}x_k t_k$ denotes their scalar product; in particular, $\mathrm{x}\cdot\mathrm{1}=\sum_{k=1}^{n}x_k$ for $\mathrm{1}=(1,1,...,1)$. The expression $\mathrm{x}>\mathrm{t}$ means that $x_1>t_1,...,x_n>t_n$; the nonstrict inequality $\geq$ has a similar meaning. We set $\mathrm{R}_+^n=\{\mathrm{x}\in\mathrm{R}^n:\mathrm{x}>0\}$ and $\mathrm{k}=(k_1,...,k_n)\in\mathrm{N}_0^n=\mathrm{N}_0\times...\times\mathrm{N}_0$, $(k_i\in\mathrm{N}_0$, where $i=1,2,...,n)$ is a multi-index with $\mathrm{k}! = k_1!\cdots k_n!$ and $|\mathrm{k}| = k_1 + k_2 + \ldots + k_n$. For $\mathrm{x} \in \mathrm{R}^n$, $\mathrm{k} \in \mathrm{N}_0^n$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathrm{R}_+^n$, we set $\mathrm{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, $\mathrm{D}^\alpha = \dfrac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} (\partial x_2)^{\alpha_2} \ldots (\partial x_n)^{\alpha_n}}$, $\Gamma(\alpha) = \Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)$.

Let $A = \|a_{jk}\|$ $(a_{jk} \in \mathrm{R}^1)$ be an $n \times n$ matrix with the determinant $|A| = \det A$; we denote its vector-rows by $\mathrm{a}_j = (a_{j1}, a_{j2}, \ldots, a_{jn})$ and the elements of the inverse matrix $A^{-1}$ by $\tilde{a}_{jk}$. Without loss of generality, we assume that $|A| = 1$. Let $A \cdot \mathrm{x} = (\mathrm{a}_1 \cdot \mathrm{x}, \mathrm{a}_2 \cdot \mathrm{x}, \ldots, \mathrm{a}_n \cdot \mathrm{x})$ and $(A \cdot \mathrm{x})^\alpha = (\mathrm{a}_1 \cdot \mathrm{x})^{\alpha_1} (\mathrm{a}_2 \cdot \mathrm{x})^{\alpha_2} \cdots (\mathrm{a}_n \cdot \mathrm{x})^{\alpha_n}$. For $\mathrm{b} = (b_1, b_2, \ldots, b_n) \in \mathrm{R}^n$, $\mathrm{c} = (c_1, c_2, \ldots, c_n) \in \mathrm{R}^n$, and $r \in \mathrm{R}^1$, by

$$A_{\mathrm{c},r}(\mathrm{b}) = \{\mathrm{t} \in \mathrm{R}^n : A \cdot (\mathrm{b} - \mathrm{t}) \geq 0,\ \mathrm{c} \cdot \mathrm{t} + r \geq 0\}, \tag{1}$$

we denote the bounded $n$ – pyramid in $\mathrm{R}^n$ with the vertex at $\mathrm{b}$, the base in the hyperplane $\mathrm{c} \cdot \mathrm{t} + r = 0$, and the lateral faces in the hyperplanes $\mathrm{a}_j \cdot (\mathrm{b} - \mathrm{t}) = 0$ $(j = 1, \ldots, n)$. In particular, if $A = E = \|\delta_{jk}\|$ is the identity matrix, $\mathrm{c} = (1,1,\ldots,1)$, and $r = 0$, then $E_1(\mathrm{b})$ is the model pyramid

$$E_1(\mathrm{b}) = \{\mathrm{t} \in \mathrm{R}^n : \mathrm{t} \leq \mathrm{b},\ \mathbf{1} \cdot \mathrm{t} \geq 0\}. \tag{2}$$

As it is known [1, Lemma 28.2], the pyramid (1) is bounded if and only if $A^{-1}\mathrm{c} \cdot \mathrm{b} > 0$ (for the pyramid (2) respectively $A^{-1}\mathrm{c} > 0$).

For $\nu = (\nu_1, \ldots, \nu_n) \in \mathrm{R}^n$ and и $\mathrm{x} = (x_1, x_2, \ldots, x_n) \in \mathrm{R}^n$ we introduce the function

$$\bar{J}_\nu[\mathrm{x}] = \prod_{j=1}^n \bar{J}_{\nu_j}[x_j], \tag{3}$$

which is the product of the Bessel – Klifford functions $\bar{J}_\nu(z)$ defined by the formula [1, § 37.1]

$$\bar{J}_\nu(z) = \Gamma(\nu + 1)\left(\frac{z}{2}\right)^{-\nu} J_\nu(z),\ |z| < \infty, \tag{4}$$

where $J_\nu(z)$ is the Bessel function of the first kind [1, § 1.3; 13, chapters 9, 10]

$$J_\nu(z) = \sum_{k=0}^\infty \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+\nu}}{\Gamma(\nu + k + 1)k!}. \tag{5}$$

The Abel-type integral equation under consideration has the form

$$\frac{1}{\Gamma(\alpha)} \int_{A_{\mathrm{c},r}(\mathrm{x})} (A \cdot (\mathrm{x} - \mathrm{t}))^{\alpha - 1}\ \bar{J}_{\frac{\alpha-1}{2}}[A \cdot \lambda(\mathrm{x} - \mathrm{t})]\ f(\mathrm{t})d\mathrm{t} = g(\mathrm{x}),\quad \mathrm{x} \in A_{\mathrm{c},r}(\mathrm{b}), \tag{6}$$

where $A_{\mathrm{c},r}(\mathrm{b})$ $(\mathrm{c}, \mathrm{b} \in \mathrm{R}^n, r \in \mathrm{R}^1)$ is a pyramid of the form (1); $\mathrm{x}, \mathrm{t}, \alpha, \lambda \in \mathrm{R}^n$, $0 < \alpha < 1$, and $\bar{J}_{\frac{\alpha-1}{2}}[A \cdot \lambda(\mathrm{x} - \mathrm{t})]$ is a function of the form (3). This equation generalizes the proper one-dimensional integral equation (see [1, § 37.1]).

We need the integral formula of the convolution type for the Bessel function (5) [12, 7.7(6)]:

$$\int_0^t \tau^\mu J_\mu(\tau)(t - \tau)^\nu J_\nu(t - \tau)d\tau = \frac{1}{\sqrt{2\pi}} \frac{\Gamma\left(\nu + \frac{1}{2}\right)\Gamma\left(\mu + \frac{1}{2}\right) t^{\nu + \mu + \frac{1}{2}}}{\Gamma(\nu + \mu + 1)} J_{\nu + \mu + \frac{1}{2}}(t),\quad \mathrm{Re}(\mu) > -\frac{1}{2},\ \mathrm{Re}(\nu) > -\frac{1}{2}, \tag{7}$$

and the following auxiliary assertion [1, Lemma 28.3].

**Lemma 1.** *If a function* $f(\mathrm{t},\boldsymbol{\tau})$ *on* $A_c(\mathrm{b})\times A_c(\mathrm{b})$ *is measurable, then the formula*

$$\int\limits_{A_c(\mathrm{b})} d\mathrm{t} \int\limits_{A_c(\mathrm{t})} f(\mathrm{t},\boldsymbol{\tau})d\boldsymbol{\tau} = \int\limits_{A_c(\mathrm{b})} d\boldsymbol{\tau} \int\limits_{\sigma(\mathrm{b},\tau)} f(\mathrm{t},\boldsymbol{\tau})d\mathrm{t}\,, \tag{8}$$

*for changing the order of integration is valid, where*

$$\sigma(\mathrm{b},\boldsymbol{\tau}) = \left\{\mathrm{t}\in \mathrm{R}^n : A\cdot\boldsymbol{\tau} \le A\cdot\mathrm{t} \le A\cdot\mathrm{b}\right\}, \tag{9}$$

*provided that one of the multiple integrals in* (8) *converges absolutely.*

**3. Solution in the closed-form.** First, we give a formal solution of the Eq. (6). Replacing in (6) $\mathrm{x}$ by $\mathrm{t}$ and $\mathrm{t}$ by $\mathrm{u}$, multiplying both sides of the resulting equality by $\left(A\cdot(\mathrm{x}-\mathrm{t})\right)^{-\alpha}\bar{J}_{-\frac{\alpha}{2}}\left[A\cdot\lambda(\mathrm{x}-\mathrm{t})\right]$, integrating over the pyramid $A_{c,r}(\mathrm{x})$, we obtain

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int\limits_{A_{c,r}(\mathrm{x})}\left(A\cdot(\mathrm{x}-\mathrm{t})\right)^{-\alpha}\bar{J}_{-\frac{\alpha}{2}}\left[A\cdot\lambda(\mathrm{x}-\mathrm{t})\right]d\mathrm{t}\int\limits_{A_{c,r}(\mathrm{t})}\left(A\cdot(\mathrm{t}-\mathrm{u})\right)^{\alpha-1}\bar{J}_{\frac{\alpha-1}{2}}\left[A\cdot\lambda(\mathrm{t}-\mathrm{u})\right]f(\mathrm{u})d\mathrm{u} = \\ = \int\limits_{A_{c,r}(\mathrm{x})}\left(A\cdot(\mathrm{x}-\mathrm{t})\right)^{-\alpha}\bar{J}_{-\frac{\alpha}{2}}\left[A\cdot\lambda(\mathrm{x}-\mathrm{t})\right]g(\mathrm{t})d\mathrm{t}\,,\ \mathrm{x}\in A_{c,r}(\mathrm{b})\,.\end{aligned} \tag{10}$$

Changing the order of integration in the left side of the (10) according to the formula (8), we obtain:

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int\limits_{A_{c,r}(\mathrm{x})} f(\mathrm{u})d\mathrm{u}\int\limits_{\sigma(\mathrm{x},\mathrm{u})}\left(A\cdot(\mathrm{x}-\mathrm{t})\right)^{-\alpha}\left(A\cdot(\mathrm{t}-\mathrm{u})\right)^{\alpha-1}\cdot\bar{J}_{\frac{\alpha-1}{2}}\left[A\cdot\lambda(\mathrm{t}-\mathrm{u})\right]\bar{J}_{-\frac{\alpha}{2}}\left[A\cdot\lambda(\mathrm{x}-\mathrm{t})\right]d\mathrm{t} = \\ = \int\limits_{A_{c,r}(\mathrm{x})}\left(A\cdot(\mathrm{x}-\mathrm{t})\right)^{-\alpha}\bar{J}_{-\frac{\alpha}{2}}\left[A\cdot\lambda(\mathrm{x}-\mathrm{t})\right]g(\mathrm{t})d\mathrm{t}\,,\end{aligned} \tag{11}$$

where $\sigma(\mathrm{x},\mathrm{u}) = \left\{\mathrm{t}\in\mathrm{R}^n : A\cdot\mathrm{u}\le A\cdot\mathrm{t}\le A\cdot\mathrm{x}\right\}$.

To calculate the inner integral in the left side of the (11), we introduce the new variables

$$s_j = \mathrm{a}_j\cdot\lambda(\mathrm{x}-\mathrm{t})\,,\ \mathrm{a}_j = \left(a_{j1},\ldots,a_{jn}\right)\ (j=1,\ldots,n)\,.$$

Using the formula (4) for the Bessel – Klifford function $\bar{J}_\nu[z]$ and the formula (7) for the inner integral in (11) we obtain

$$\frac{1}{\Gamma(\alpha)}\int\limits_{\sigma(\mathrm{x},\mathrm{u})}\left(A\cdot(\mathrm{x}-\mathrm{t})\right)^{-\alpha}\left(A\cdot(\mathrm{t}-\mathrm{u})\right)^{\alpha-1}\bar{J}_{\frac{\alpha-1}{2}}\left[A\cdot\lambda(\mathrm{t}-\mathrm{u})\right]\bar{J}_{-\frac{\alpha}{2}}\left[A\cdot\lambda(\mathrm{x}-\mathrm{t})\right]d\mathrm{t} =$$

$$= \frac{1}{\Gamma(\alpha)}\prod_{j=1}^{n}\left[\int\limits_0^{\mathrm{a}_j\cdot\lambda(\mathrm{x}-\mathrm{u})} s_j^{-\alpha_j}\bar{J}_{-\frac{\alpha}{2}}\left[s_j\right]\left(\mathrm{a}_j\cdot\lambda(\mathrm{x}-\mathrm{u})-s_j\right)^{\alpha_j-1}\bar{J}_{\frac{\alpha_j-1}{2}}\left[\mathrm{a}_j\cdot\lambda(\mathrm{x}-\mathrm{u})-s_j\right]ds_j\right] =$$

$$\begin{aligned}= \frac{1}{\Gamma(\alpha)}\prod_{j=1}^{n}\Bigg[\Gamma\left(1-\frac{\alpha_j}{2}\right)\Gamma\left(\frac{\alpha_j+1}{2}\right)\int\limits_0^{\mathrm{a}_j\cdot\lambda(\mathrm{x}-\mathrm{u})} s_j^{-\alpha_j}\left(\frac{s_j}{2}\right)^{\frac{\alpha_j}{2}} J_{-\frac{\alpha_j}{2}}\left(s_j\right)\times \\ \times\left(\mathrm{a}_j\cdot\lambda(\mathrm{x}-\mathrm{u})-s_j\right)^{\alpha_j-1}\left(\frac{\left(\mathrm{a}_j\cdot\lambda(\mathrm{x}-\mathrm{u})-s_j\right)}{2}\right)^{\frac{1-\alpha_j}{2}} J_{\frac{\alpha_j-1}{2}}\left(\mathrm{a}_j\cdot\lambda(\mathrm{x}-\mathrm{u})-s_j\right)ds_j\Bigg] =\end{aligned}$$

$$=\frac{\Gamma\left(\mathbf{1}-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+\mathbf{1}}{2}\right)}{\sqrt{\mathbf{2}}\,\Gamma(\alpha)}\prod_{j=1}^{n}\left[\int_{0}^{a_j\cdot\lambda(x-u)} s_j^{-\frac{\alpha_j}{2}} J_{-\frac{\alpha_j}{2}}(s_j)\cdot\left(a_j\cdot\lambda(x-u)-s_j\right)^{\frac{\alpha_j-1}{2}} J_{\frac{\alpha_j-1}{2}}\left(a_j\cdot\lambda(x-u)-s_j\right)ds_j\right]=$$

$$=\frac{\Gamma\left(\mathbf{1}-\frac{\alpha}{\mathbf{2}}\right)\Gamma\left(\frac{\alpha+\mathbf{1}}{\mathbf{2}}\right)}{\sqrt{\mathbf{2}}\,\Gamma(\alpha)}\frac{1}{\sqrt{\mathbf{2}\pi}}\frac{\Gamma\left(\frac{\mathbf{1}-\alpha}{\mathbf{2}}\right)\Gamma\left(\frac{\alpha}{\mathbf{2}}\right)}{\Gamma\left(\frac{\mathbf{1}}{\mathbf{2}}\right)}\cdot t^{-\frac{\alpha}{2}+\frac{\alpha}{2}-\frac{1}{2}+\frac{1}{2}}\cdot J_{-\frac{\alpha}{2}+\frac{\alpha}{2}-\frac{1}{2}+\frac{1}{2}}\left(A\cdot\lambda(x-u)\right)=$$

$$=\frac{\boldsymbol{\pi}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)}{2\boldsymbol{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\boldsymbol{\pi}}{2}\right)}\cdot J_0\left(A\cdot\lambda(x-u)\right)=\frac{\Gamma\left(\frac{\alpha+\mathbf{1}}{2}\right)\Gamma\left(\frac{\mathbf{1}-\alpha}{2}\right)}{2\,\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)}\cdot J_0\left(A\cdot\lambda(x-u)\right),$$

where $\mathbf{2}=(2,...,2)$, $\boldsymbol{\pi}=(\pi,...,\pi)$.

Thus, the equality (11) takes the form:

$$\int_{A_{c,r}(x)} J_0\left(A\cdot\lambda(x-t)\right) f(t)dt = f_{A_{c,r}}^{\lambda,\alpha}(x), \tag{12}$$

or

$$\int_{A_{c,r}(x)} f^*(t)dt = f_{A_{c,r}}^{\lambda,\alpha}(x),$$

where

$$f^*(t)=J_0\left(A\cdot\lambda(x-t)\right) f(t),$$

$$f_{A_{c,r}}^{\lambda,\alpha}(x)=\frac{\mathbf{2}\,\Gamma(\alpha)\sin\left(\frac{\alpha\boldsymbol{\pi}}{\mathbf{2}}\right)}{\Gamma\left(\frac{\alpha+\mathbf{1}}{\mathbf{2}}\right)\Gamma\left(\frac{\mathbf{1}-\alpha}{\mathbf{2}}\right)}\int_{A_{c,r}(x)}\left(A\cdot(x-t)\right)^{-\alpha}\bar{J}_{-\frac{\alpha}{\mathbf{2}}}\left[A\cdot\lambda(x-t)\right]g(t)dt. \tag{13}$$

Making the change of variables

$$x+\frac{r}{nc}=A^{-1}\cdot\frac{y}{d},\ t+\frac{r}{nc}=A^{-1}\cdot\left(\frac{\boldsymbol{\tau}}{d}\right), \tag{14}$$

where $\frac{y}{d}=\left(\frac{y_1}{d_1},...,\frac{y_n}{d_n}\right)\in R^n$ and $d=A^{-1}\cdot c$, we represent (12) in the form

$$\int_{E_1(y)}\psi(\boldsymbol{\tau})d\boldsymbol{\tau}=\varphi(y), \tag{15}$$

where $E_1(y)$ is the model pyramid (2) and

$$\psi(\boldsymbol{\tau}) = f^*\left(A^{-1}\cdot\left(\frac{\boldsymbol{\tau}}{d}\right)-\frac{r}{n\,c}\right),\ \varphi(y)=f_{A_{c,r}}^{\lambda,\alpha}\left(A^{-1}\cdot\left(\frac{y}{d}\right)-\frac{r}{nc}\right)\prod_{j=1}^{n}d_j.$$

To invert Eq. (15), we rewrite this equation in the form

$$\int_{-(y_1+...+y_{n-1})}^{y_n}d\tau_n\int_{-(y_1+...+y_{n-2}+\tau_n)}^{y_{n-1}}d\tau_{n-1}...\int_{-(\tau_2+...+\tau_n)}^{y_1}\psi(\boldsymbol{\tau})d\tau_1=\varphi(y). \tag{16}$$

Successively differentiating with respect to $y_n, y_{n-1},..., y_1$, we obtain

$$\psi(y)=\frac{\partial}{\partial y}\varphi(y)=\frac{\partial}{\partial y_1}...\frac{\partial}{\partial y_n}\varphi(y).$$

Returning to the variable $x = A^{-1} \cdot \frac{y}{d} - \frac{r}{n c}$ in (14) and taking into account the equalities

$$\frac{\partial}{\partial y_k} = \sum_{j=1}^{n} \frac{\tilde{a}_{jk}}{d_k} \frac{\partial}{\partial y_j} \quad (k = 1, ..., n), \tag{17}$$

where the $\tilde{a}_{jk}$ $(j, k = 1, ..., n)$ are the elements of the inverse matrix $A^{-1}$, we arrive at the following formula for the solution of Eq. (15) and taking into consideration that $J_0(A \cdot \lambda(x - x)) = J_0(0) = 1$, we come at the following form of the equation solution (6):

$$f(x) = \prod_{k=1}^{n} \left( \sum_{j=1}^{n} \tilde{a}_{jk} \frac{\partial}{\partial x_j} \right) \left\{ \frac{2\,\Gamma(\alpha) \sin\left(\frac{\alpha\pi}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)} \int_{A_{c,r}(x)} (A \cdot (x - t))^{-\alpha} \bar{J}_{-\frac{\alpha}{2}} [A \cdot \lambda(x - t)]\, g(t) dt \right\}. \tag{18}$$

Thus, we have proved that if the Eq. (6) is solvable, then its solution has the form (18).

**4. Necessary and sufficient solvability conditions.** Let us prove the necessary and sufficient solvability conditions for the Eq. (6) in the space $L_1(A_{c,r}(b))$ defined by

$$L_1(A_{c,r}(b)) = \left\{ f(x) : \int_{A_{c,r}(x)} |f(t)| dt < \infty \right\}. \tag{19}$$

Consider the space

$$I_{A_{c,r}}(L_1) = \left\{ \varphi : \varphi(x) = \int_{A_{c,r}(x), A \cdot (b - t) \geq A \cdot (x - t)} h(t) dt,\ h(t) \in L_1(A_{c,r}(b)) \right\}. \tag{20}$$

The space $I_{A_{c,r}}(L_1)$ plays the same role for the Eq. (6) as the space $AC([a,b])$ of absolutely continuous functions plays for the classical Abel integral equation [1, Section 2.2]. Note that if $\varphi \in I_{A_{c,r}}(L_1)$, then, almost everywhere on $A_{c,r}(b)$, this function has partial derivatives, and

$$\prod_{k=1}^{n} \left( \sum_{j=1}^{n} \tilde{a}_{jk} \frac{\partial}{\partial x_j} \right) \varphi(x) = h(x).$$

In particular, if $A = E$ is the identity matrix, $c = \mathbf{1} = (1, ..., 1)$, and $r = 0$, then relations (19)–(20), respectively take the forms

$$L_1(E_1(b)) = \left\{ f(x) : \int_{E_1(x)} |f(t)| dt < \infty \right\},$$

$$I_{E_1}(L_1) = \left\{ \varphi : \varphi(x) = \int_{E_1(x), (b - t) \geq (x - t)} h(t) dt,\ h(t) \in L_1(E_1(b)) \right\},$$

where $$h(x) = \frac{\partial}{\partial x} \varphi(x) \equiv \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} \varphi(x).$$

Tamarkin's classical theorem on the solvability of the one-dimensional Abel integral equation in $L_1(a,b)$ has the following analogue.

THEOREM 1. *The multidimensional Abel-type integral equation (6)* $\lambda, \alpha \in \mathrm{R}^n$ $(0 < \alpha < 1)$ *is solvable in the space* $L_1\left(A_{c,r}(\mathrm{b})\right)$ *if and only if*

$$f_{A_{c,r}}^{\lambda,\alpha}(\mathrm{x}) = \frac{2\,\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)} \int\limits_{A_{c,r}(\mathrm{x})} \left(A\cdot(\mathrm{x}-\mathrm{t})\right)^{-\alpha} \bar{J}_{-\frac{\alpha}{2}}\left[A\cdot\lambda(\mathrm{x}-\mathrm{t})\right] g(\mathrm{t})d\mathrm{t} \in I_{A_{c,r}}\left(L_1\right), \tag{21}$$

$$\left[f_{A_{c,r}}^{\lambda,\alpha}(\mathrm{x})\right]_{c\cdot \mathrm{x}+r=0} = \left[\sum_{j=1}^{n} \tilde{a}_{jk} \frac{\partial}{\partial x_j} f_{A_{c,r}}^{\lambda,\alpha}(\mathrm{x})\right]_{c\cdot \mathrm{x}+r=0} = \ldots = \left[\prod_{k=2}^{n}\sum_{j=1}^{n}\left(\tilde{a}_{jk}\frac{\partial}{\partial x_j}\right) f_{A_{c,r}}^{\lambda,\alpha}(\mathrm{x})\right]_{c\cdot \mathrm{x}+r=0} = 0. \tag{22}$$

*Under these conditions, the Eq.(6) is uniquely solvable in* $L_1\left(A_{c,r}(\mathrm{b})\right)$ *and its solution is given by (18).*

***Proof.*** In the model case $A_{c,r}(\mathrm{b}) = E_1(\mathrm{b})$, the required assertion follows from (15) and (16). In the case of an arbitrary pyramid $A_{c,r}(\mathrm{b})$, this assertion is obtained from (15) and (16) by making the change of variables (14) and taking into account (17).

**Corollary 1.** *The multidimensional Abel-type model integral equation*

$$\frac{1}{\Gamma(\alpha)} \int\limits_{E_1(\mathrm{x})} (\mathrm{x}-\mathrm{t})^{\alpha-1}\, \bar{J}_{\frac{\alpha-1}{2}}\left[\lambda(\mathrm{x}-\mathrm{t})\right] f(\mathrm{t})d\mathrm{t} = g(\mathrm{x}), \quad \mathrm{x}\in E_1(\mathrm{b}), \tag{23}$$

$\lambda, \alpha \in \mathrm{R}^n$ $(0 < \alpha < 1)$, *is solvable in the space* $L_1\left(E_1(\mathrm{b})\right)$ *if and only if*

$$f_{E_1}^{\lambda,\alpha}(\mathrm{x}) = \frac{2\,\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)} \int\limits_{E_1(\mathrm{x})} (\mathrm{x}-\mathrm{t})^{-\alpha}\, \bar{J}_{-\frac{\alpha}{2}}\left[\lambda(\mathrm{x}-\mathrm{t})\right] g(\mathrm{t})d\mathrm{t} \in I_{E_1}\left(L_1\right)$$

*and*

$$\left[f_{E_1}^{\lambda,\alpha}(\mathrm{x})\right]_{1\cdot \mathrm{x}=0} = \left[\frac{\partial}{\partial x_n} f_{E_1}^{\lambda,\alpha}(\mathrm{x})\right]_{1\cdot \mathrm{x}=0} = \ldots = \left[\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n} f_{E_1}^{\lambda,\alpha}(\mathrm{x})\right]_{1\cdot \mathrm{x}=0} = 0.$$

*Under these conditions, Eq.* (23) *is uniquely solvable in* $L_1\left(E_1(\mathrm{b})\right)$, *and its solution is given by*

$$f(\mathrm{x}) = \frac{\partial}{\partial \mathrm{x}} f_{E_1}^{\lambda,\alpha}(\mathrm{x}) = \frac{\partial}{\partial \mathrm{x}}\left\{\frac{2\,\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)} \int\limits_{E_1(\mathrm{x})} (\mathrm{x}-\mathrm{t})^{-\alpha} \bar{J}_{-\frac{\alpha}{2}}\left[\lambda(\mathrm{x}-\mathrm{t})\right] g(\mathrm{t})d\mathrm{t}\right\}.$$

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## РЕШЕНИЕ МНОГОМЕРНОГО ИНТЕГРАЛЬНОГО УРАВНЕНИЯ ПЕРВОГО РОДА С ФУНКЦИЕЙ БЕССЕЛЯ – КЛИФФОРДА В ЯДРЕ ПО ПИРАМИДАЛЬНОЙ ОБЛАСТИ

***канд. физ.-мат. наук, доц. О.В. СКОРОМНИК, Т.А. АЛЕКСАНДРОВИЧ***
***(Полоцкий государственный университет)***

*Рассматривается многомерное интегральное уравнение первого рода с функцией Бесселя – Клиффорда в ядре по ограниченной пирамидальной области многомерного евклидова пространства специального вида. Интерес к исследованию таких уравнений вызван их приложениями в задачах исследования отражения волн от прямолинейной границы и в задачах сверхзвукового обтекания пространственных углов. Хорошо известен классический результат Я. Тамаркина о разрешимости интегрального уравнения Абеля в пространстве* $L_1(a,b)$ *суммируемых функций на конечном отрезке* $[a,b]$ *действительной оси. Следуя методике Я. Тамаркина, устанавливается формула решения исследуемого уравнения в замкнутой форме, даются необходимые и достаточные условия его разрешимости в пространстве суммируемых функций. Доказанные утверждения обобщают результаты, полученные ранее для многомерного уравнения типа Абеля и для соответствующих одномерных гипергеометрических уравнений.*